Hunters Hill High School

Extension 2, Mathematics

Trial Examination, 2018

Hunters Hill
High School

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- The marks for each question are shown on the paper
- Show relevant mathematical reasoning and/or calculations in Questions 11-16

Total Marks: 100

Section I Page 3 – 6
10 marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 7 – 13
90 marks
- Attempt Questions 11-16
- Begin a **new sheet** for each question
- Allow about 2 hour 45 minutes for this section

Section I

Total marks – 10
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1. Which of the following ellipses has focus $(1, 0)$ and directrix $x = 4$?

(A) $\frac{x^2}{3} + \frac{y^2}{4} = 1$

(B) $4x^2 + 9y^2 = 36$

(C) $3x^2 + 4y^2 = 12$

(D) $\frac{x^2}{4} + \frac{y^2}{\sqrt{3}} = 1$

2. Find Arg z_1 where $z_1 = \left(-1 + \sqrt{3}i\right)^8$

(A) $\frac{16\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) $-\frac{2\pi}{3}$

(D) $-\frac{\pi}{3}$

3. $P(x)$ is a polynomial of degree 4 with real coefficients. $P(x)$ has $x = 2$ as a root of multiplicity 2, and $x = -i$ as a root.

Which of the following expressions is a factorised form of $P(x)$ over the complex numbers?

(A) $P(x) = (x-2)^2(x-1)(x+1)$

(B) $P(x) = (x+2)^2(x-1)^2$

(C) $P(x) = (x-2)^2(x-i)(x+i)$

(D) $P(x) = (x+2)^2(x-i)(x+i)$

4. The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$.

Which of the following expressions is correct?

(A) $\tan\theta \,.\, \tan\phi = \dfrac{a^2}{b^2}$

(B) $\tan\theta \,.\, \tan\phi = -\dfrac{a^2}{b^2}$

(C) $\tan\theta \,.\, \tan\phi = \dfrac{b^2}{a^2}$

(D) $\tan\theta \,.\, \tan\phi = -\dfrac{b^2}{a^2}$

5. Find $\displaystyle\int x\ln x\, dx$

(A) $1 - x\ln x + c$

(B) $1 - \displaystyle\int \ln x\, dx + c$

(C) $\dfrac{1}{2}x^2\ln x - \dfrac{1}{4}x^2 + c$

(D) $\dfrac{1}{2}x^2\ln x - \dfrac{1}{4}x^2 + c$

6. A particle of mass m is undergoing circular motion in a circle of radius r with angular velocity ω. Let the particle's tangential velocity be V and let its tangential acceleration be a.

Which of the following expressions is correct?

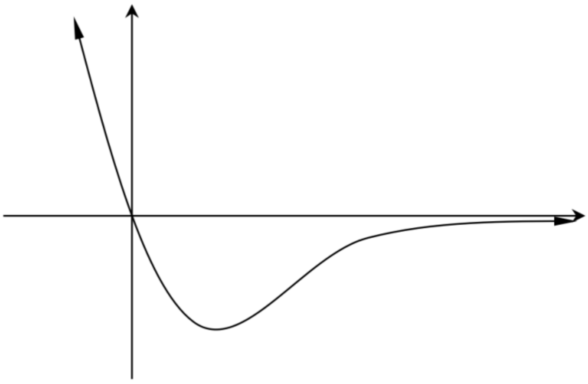
(A) $V = r\omega$

(B) $V = mr\omega$

(C) $a = r\omega^2$

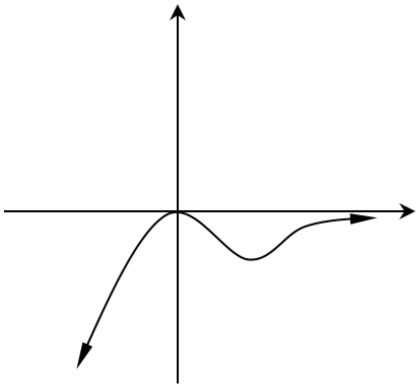
(D) $a = mr\omega^2$

7. The diagram shows a sketch of the curve $y = x\,f(x)$.

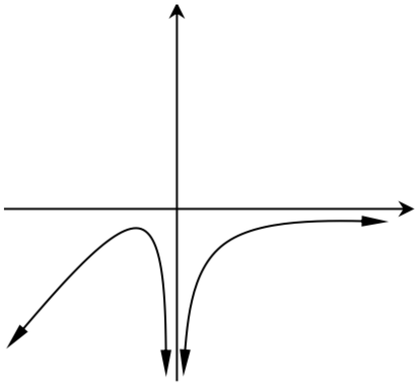


Which of the following best represents $y = f(x)$.

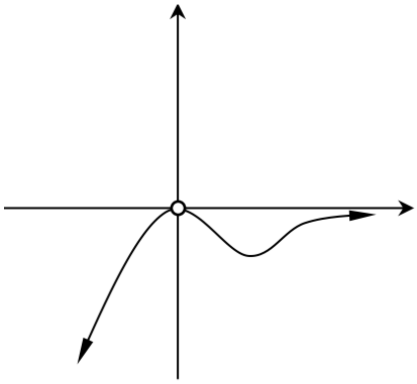
(A)



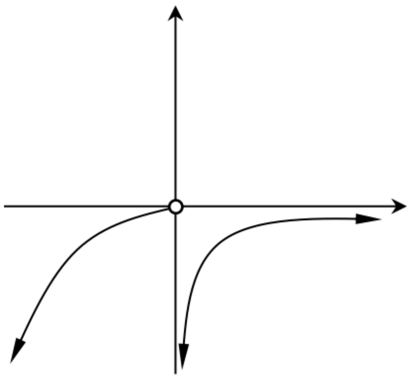
(B)



(C)



(D)



8. Find $\int \cos x \sin^5 x \, dx$

(A) $\dfrac{1}{6}\cos^6 x$

(B) $-\dfrac{1}{6}\cos x \sin^7 x$

(C) $6\sin^6 x$

(D) $\dfrac{1}{6}\sin^6 x$

9. A particle of mass m is moving in a straight line under the action of a force

$$F = \frac{m}{x^3}(6 - 10x)$$

Which of the following is an expression for it's velocity in any position, if the particle starts from rest at $x = 1$?

(A) $v = \pm\frac{1}{x}\sqrt{-6 + 5x + x^2}$

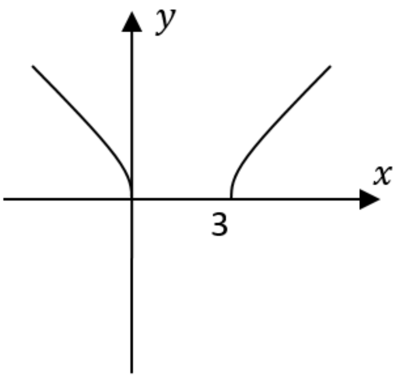
(B) $v = \pm\frac{2}{x}\sqrt{-6 + 5x + x^2}$

(C) $v = \pm x\sqrt{-6 + 5x + x^2}$

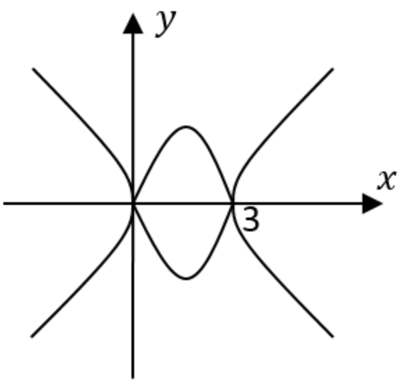
(D) $v = \pm 2x\sqrt{-6 + 5x + x^2}$

10. If $f(x) = x(x-3)$, which of the following graphs best represents the curve $y^2 = f(x)$

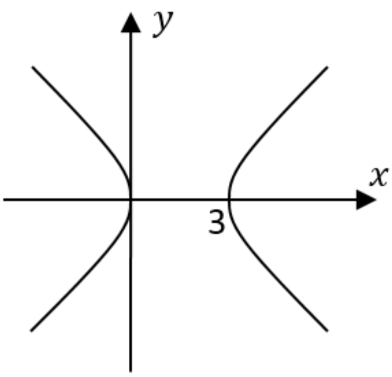
(A)



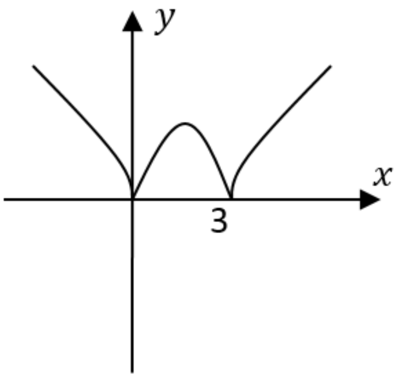
(B)



(C)



(D)



End of Section I

Section II

Total marks – 90
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Begin each question on a NEW sheet of paper.

Question 11 (15 marks) Use a NEW sheet of paper.

a. Find $\int \frac{dx}{x^2 + 2x + 3}$ **2**

b. Find $\int \cos^5 x \, dx$ **3**

c. Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate **4**

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1 + \cos x}$$

d. Find the equation of the tangent to the curve $2x^2 + 2xy - y^2 + 1 = 0$ at the point $(1, 3)$. **3**

e. Sketch, on separate Argand diagrams, the locus of all points z such that:

i. $\arg z = \frac{\pi}{6}$ **1**

ii. $\arg \bar{z} = \frac{\pi}{6}$ **1**

iii. $\arg(-z) = \frac{\pi}{6}$ **1**

End of Question 11

Question 12 (15 marks) Use a NEW sheet of paper.

a. If $z = 5 - 2i$, find in the form $x + iy$:

i. z^2 **1**

ii. $z + 2\bar{z}$ **1**

iii. $\frac{i}{z}$ **2**

b. The roots of the polynomial equation $2x^3 - 3x^2 + 4x - 5 = 0$ are α, β and γ.
Find the polynomial equation that has roots:

i. $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ **2**

ii. $2\alpha, 2\beta$ and 2γ **2**

c. Find the exact values of x and y, such that $(x + iy)^2 = 7 - 24i$, where x and y are real. **3**

d.

i. Find real numbers a and b such that **2**

$$\frac{x-4}{x^2+5x+4} = \frac{a}{x+4} + \frac{b}{x+1}$$

ii. Hence evaluate **2**

$$\int_2^4 \frac{x-4}{x^2+5x+4}\,dx$$

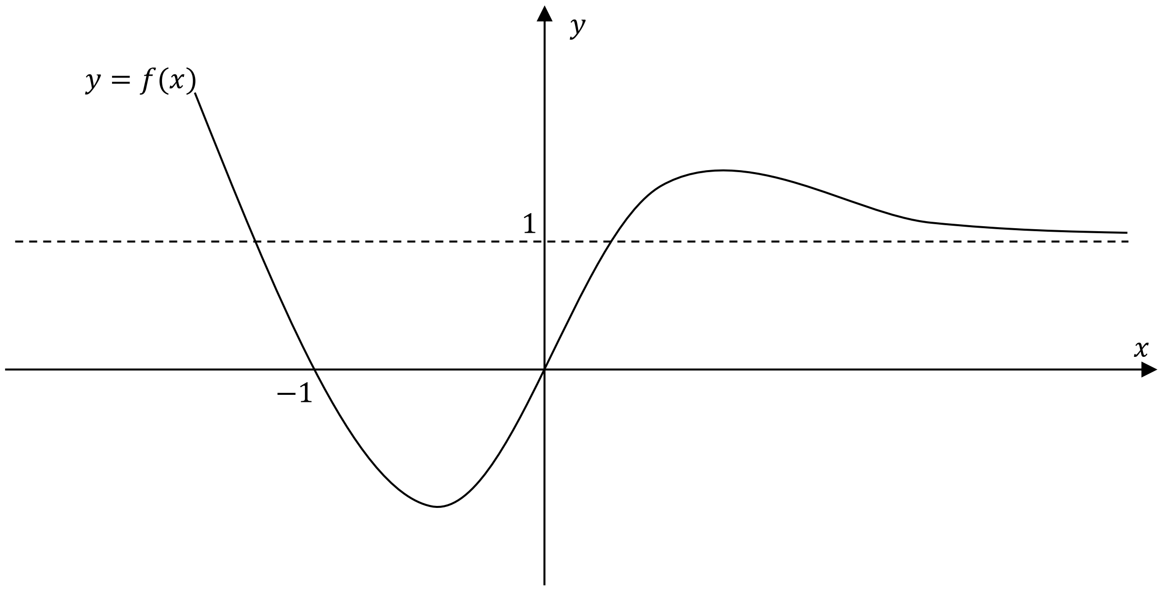
End of Question 12

Question 13 (15 marks) Use a NEW sheet of paper.

a. The base of a solid is the segment of the parabola $x^2 = 4y$ cut off by the line $y = 3$. Cross-sections taken perpendicular to the axis of the parabola are rectangle, whose heights are twice their base.

Find the volume of the solid. **3**

b. The diagram shows the graph of $y = f(x)$.



Draw separate **one-third** page sketches of the graphs of the following:

i. $y = |f(x)|$ **1**

ii. $y^2 = f(x)$ **2**

c. Find the cube roots of $1 - i$. **4**

d. **i.** $I_n = \int x^n e^{ax} dx$, where a is a constant. **2**

Prove that $I_n = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} I_{n-1}$.

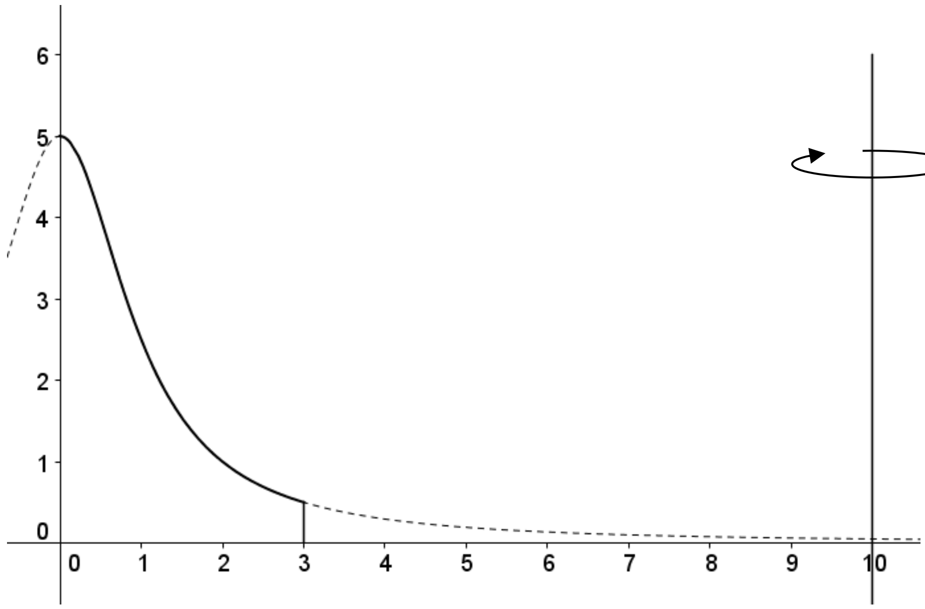
ii. Hence find the value of $\int_0^1 x^3 e^{2x} dx$. **3**

End of Question 13

Question 14 (15 marks) Use a NEW sheet of paper.

a. A circular flange is formed by rotating the region bounded by the curve $y = \dfrac{5}{x^2+1}$, the x-axis and the lines $x = 0$, and $x = 3$, through one complete revolution about the line $x = 10$.

(All measurements are in centimetres)



i. Use the method of cylindrical shells to show that the volume V cm^3 of the flange is given by **2**

$$V = 10\pi \int_0^3 \frac{10 - x}{x^2+1}\,dx$$

ii. Find the volume of the flange correct to the nearest cm^3. **3**

b. $P\left(3p, \dfrac{3}{p}\right)$ and $Q\left(3q, \dfrac{3}{q}\right)$ are points on different branches of the hyperbola $xy = 9$.

i. Find the equation of the tangent at P. **2**

ii. Find the point of intersection, T, of the tangents at P and Q. **2**

iii. If the chord PQ passes through the point $(0, 2)$, find the locus of T. **3**

iv. Find a restriction on the locus of T. **1**

c. On an Argand diagram, sketch the region described by the inequality **2**

$$\left|1 + \frac{1}{z}\right| \leq 1.$$

End of Question 14

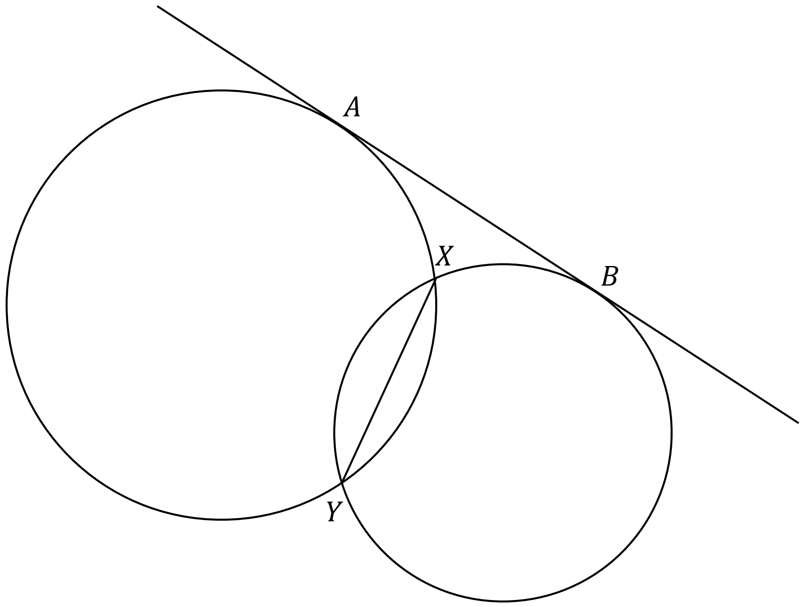
Question 15 (15 marks) Use a NEW sheet of paper.

a. The polynomial $P(x) = x^5 + 2x^2 + mx + n$ has a double zero at $x = -2$. **3**

Find the product of the other three zeros.

b. In the diagram below, AB is a common tangent and XY is a common chord to the two circles.

Extend BX to meet AY at Q and extend AX to meet BY at P.



i. Copy the diagram onto your answer sheet, showing all the information given. **1**

ii. Prove that $PXQY$ is a cyclic quadrilateral. **3**

iii. Prove that AB is parallel to PQ. **2**

iv. Prove that XY bisects PQ. **3**

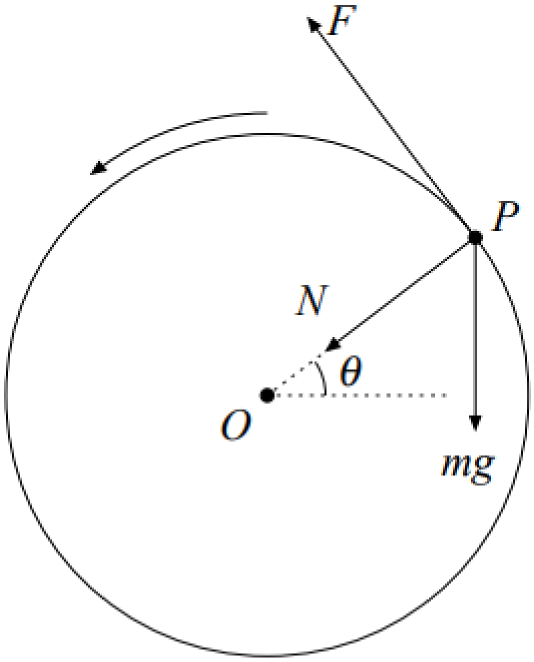
Question 15 is continued on the next page

c. A circular drum is rotating with uniform angular velocity round a horizontal axis. A particle P is rotating in a vertical circle, without slipping, on the inside of the drum.

The radius of the drum is r metres and its angular velocity is ω radians/second. Acceleration due to gravity is g metres/second^2, and the mass of P is m kilograms.

The centre of the drum is O, and OP makes an angle θ to the horizontal.

The drum exerts a normal force N on P, as well as a frictional force F, acting tangentially to the drum, as shown in the diagram.



By resolving force perpendicular to, and parallel to, OP, find an expression for $\frac{F}{N}$ in terms of the data. **3**

End of Question 15

Question 16 (15 marks) Use a NEW sheet of paper.

a. A particle is fired vertically upwards with an initial velocity, V metres per second.

The particle is subject to air resistance proportional to its speed and a downward gravitational force, g.

The equation of motion of the particle is $\ddot{x} = -g - kv$, where $k > 0$.

Show that the particle reaches a maximum height, H, given by: **5**

$$H = \frac{V}{k} - \frac{g}{k^2}\ln\left(1 + \frac{kV}{g}\right)$$

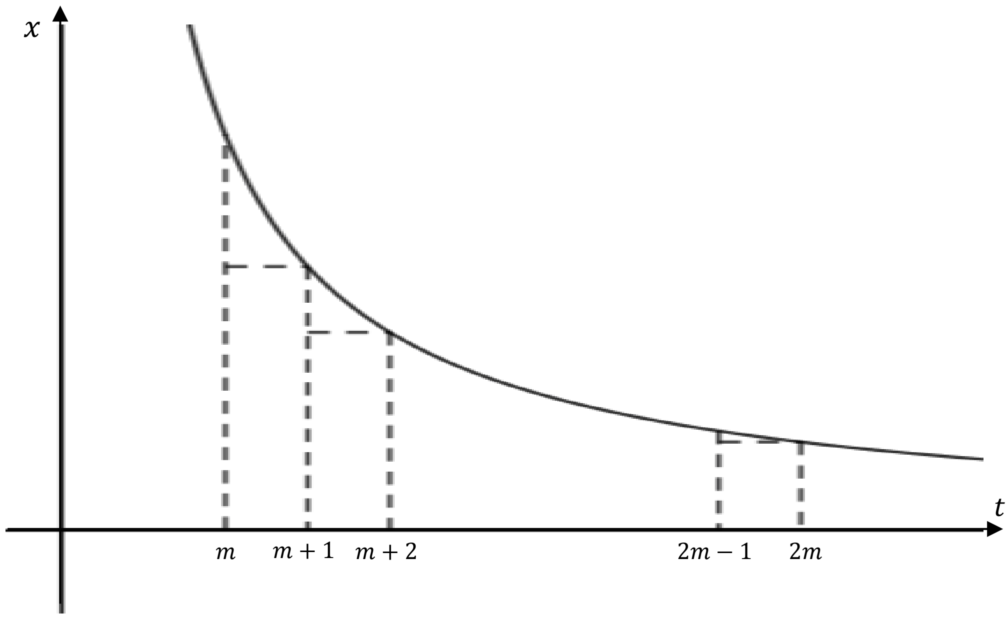
b. **i.** Prove that $\frac{1}{2p+1} + \frac{1}{2p+2} > \frac{1}{p+1}$, for all $p > 0$ **2**

ii. Consider the statement

$$\psi(m): \frac{1}{(m+1)} + \frac{1}{m+2} + \cdots + \frac{1}{2m} \geq \frac{37}{60}$$

Show, by mathematical induction, that $\psi(m)$ is true for all integers $m \geq 3$. **3**

iii. The diagram below shows the graph of $x = \frac{1}{t}$, for $t > 0$.



By comparing areas, show that $\int_m^{m+1} \frac{1}{t}\,dt > \frac{1}{m+1}$. **2**

iv. Hence, without using a calculator, show that $\log_e 2 > \frac{37}{60}$. **3**

End of paper

MXX TRIAL 2018 – HHHS

1 C

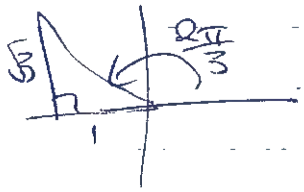
$ae = 1$, $\frac{a}{e} = 4$

$a^2 = 4$

$a = 2 \Rightarrow e = \frac{1}{2}$ $\quad \frac{1}{4} = 1 - \frac{b^2}{4}$

$b^2 = 3$

2 C.



$\frac{2\pi}{3} \times 8 = \frac{16\pi}{3}$

3. C $\quad (x-2)^2(x-i)(x+i)$.

4. B

$\frac{b\sin\theta - 0}{a\cos\theta - 0} \cdot \frac{b\sin\phi - 0}{a\cos\phi - 0} = -1$.

$\tan\theta \cdot \tan\phi = -\frac{a^2}{b^2}$

5. C/D

$y = x\ln x$

$u = \ln x \qquad dv = x\,dx$

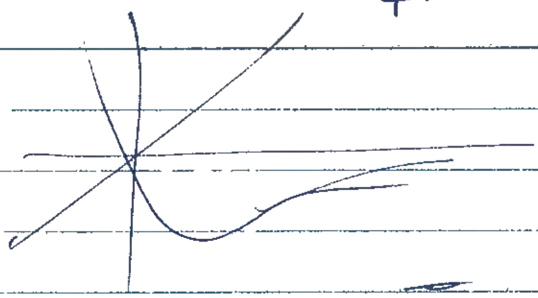
$du = \frac{1}{x}dx \qquad v = \frac{x^2}{2}$

$I = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$

$- \frac{x^2}{4}$

6. A

7 C



8 D

$u = \sin x$
$du = \cos x \, dx$

$$\int u^5 \, du = \frac{u^6}{6}$$

9. B.

$$F = \frac{M}{x^3}(6 - 10x)$$

$$a = \frac{6-10x}{x^3}$$

$$\int v \frac{dv}{dx} = \int \frac{6-10x}{x^3} \, dx$$

$$\frac{v^2}{2} = \int 6x^{-3} - 10x^{-2} \, dx$$

$$= \frac{-12}{x^2} + \frac{10}{x} + c$$

$$0 = -12 + 10 + c \Rightarrow c = 2$$

$$\frac{v^2}{2} = \frac{-12}{x^2} + \frac{10}{x} + 2$$

$$v^2 = \frac{2}{x^2}(-12 + 10x + 2x^2)$$

$$= \frac{4}{x^2}(-6 + 5x + x^2)$$

$$v = \pm \frac{2}{x}\sqrt{\ldots}$$

10 C

11 a $I=\int \frac{dx}{x^2+2x+3}$

$$=\int \frac{dx}{(x+1)^2+2}$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{(x+1)}{\sqrt{2}}+c$$

b. $\int \cos^5 x\,dx = \int \cos^4 x\,\cos x\,dx$

$$=\int (1-\sin^2 x)^2 \cos x\,dx$$

$u=\sin x$
$du=\cos x\,dx$

$$=\int (1-u^2)^2\,du$$

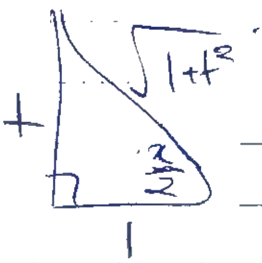
$$=\int (1-2u^2+u^4)\,du$$

$$=u-\frac{2u^3}{3}+\frac{u^5}{5}+c$$

$$=\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+c.$$

c. $t=\tan\frac{x}{2}$

$\cos x=\frac{1-t^2}{1+t^2}$



$dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dx=\frac{2dt}{}\cos^2\frac{x}{2}$

$=\frac{2\,dt}{1+t^2}$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1+\cos x}$$

$$=2\int_0^{\tan\frac{\pi}{8}} \frac{1}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{dt}{1+t^2}$$

$$=2\int_0^{\tan\frac{\pi}{8}} \frac{\cancel{1+t^2}}{2}\cdot\frac{dt}{\cancel{1+t^2}}$$

$$=\left[t\right]_0^{\tan\frac{\pi}{8}}=\tan\frac{\pi}{8}$$

d. $2x^2+2xy-y^2+1=0$

$$\frac{d}{dx} \quad 4x+2x\frac{dy}{dx}+2y-2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(2x-2y)=-(4x+2y)$$

$$\frac{dy}{dx}=\frac{-(4x+2y)}{2x-2y}$$

$$=\frac{-(2x+y)}{x-y}$$

at $(1,3)$

$$m=\frac{-(2(1)+3)}{1-3}$$

$$=\frac{5}{2}$$

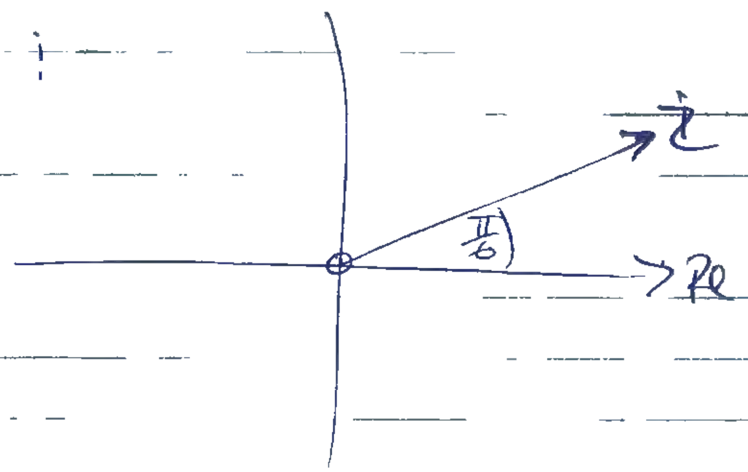
by point-gradient

$$y-3=\frac{5}{2}(x-1)$$

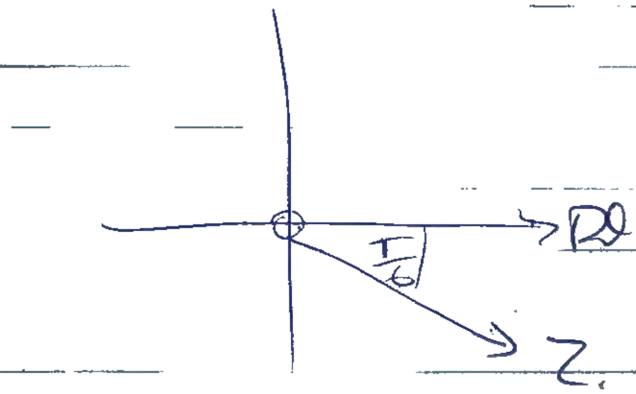
$$2y-6=5x-5$$

$\therefore 5x-2y+1=0$ is tangent.

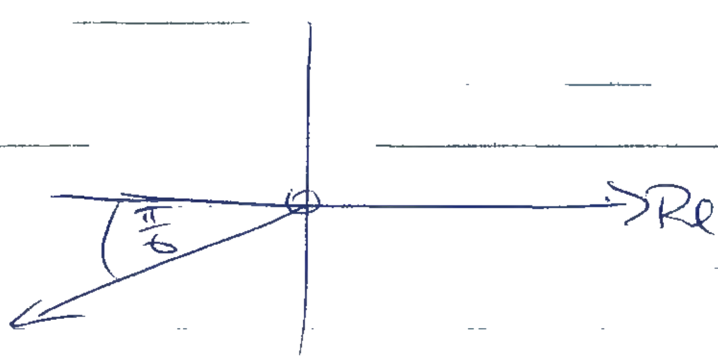
e)



ii



iii)



12 a $z = 5 - 2i$

i. $z^2 = (5-2i)^2$

$= 25 - 20i - 4$

$= 21 - 20i$

ii. $z + 2\bar{z} = 5 - 2i + 2(5+2i)$

$= 15 + 2i$

iii $\frac{i}{z} = \frac{i}{5-2i} \times \frac{5+2i}{5+2i}$

$= \frac{-2+5i}{25+4}$

$= -\frac{2}{29} + \frac{5i}{29}$

12b $2x^3-3x^2+4x-5=0$

i) $P(\frac{1}{x}): 2(\frac{1}{x})^3-3(\frac{1}{x})^2+4(\frac{1}{x})-5=0$

$$\frac{2}{x^3}-\frac{3}{x^2}+\frac{4}{x}-5=0$$

$\times x^3$: $2-3x+4x^2-5x^3=0$

$\therefore 5x^3-4x^2+3x-2=0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

ii) $P(\frac{x}{2}): 2(\frac{x}{2})^3-3(\frac{x}{2})^2+4(\frac{x}{2})-5=0$

$$\frac{x^3}{4}-\frac{3x^2}{4}+2x-5=0$$

$\therefore x^3-3x^2+8x-20=0$ has roots $2\alpha, 2\beta, 2\gamma$.

c) $(x+iy)^2=7-24i$

$x^2-y^2+2xyi=7-24i$

equating Re & Im

$x^2-y^2=7$

$\times x^2$ $\quad x^4-x^2y^2=7x^2 \quad$ — i

$2xy=-24 \quad$ — ii

subst ii into i

$x^4-(\frac{-24}{2})^2=7x^2$

$x^4-7x^2-144=0$

$(x^2-16)(x^2+9)=0$

$\therefore x=\pm 4 \quad$ as $x^2\neq -9$

$y=\frac{-12}{(\pm 4)}$

$=\mp 3$

$\therefore x=4, y=-3$ and $x=-4, y=3$ are solutions

d. i. $\dfrac{x-4}{x^2+5x+4} = \dfrac{a}{x+4} + \dfrac{b}{x+1}$

$\therefore \quad x-4 = a(x+1) + b(x+4)$

for $x=-1$

$-5 = a(0) + 3b \Rightarrow b = -\dfrac{5}{3}$

for $x=-4$

$-8 = -3a + b(0) \Rightarrow a = \dfrac{8}{3}$

$\therefore \quad a = \dfrac{8}{3},\ b = -\dfrac{5}{3}$

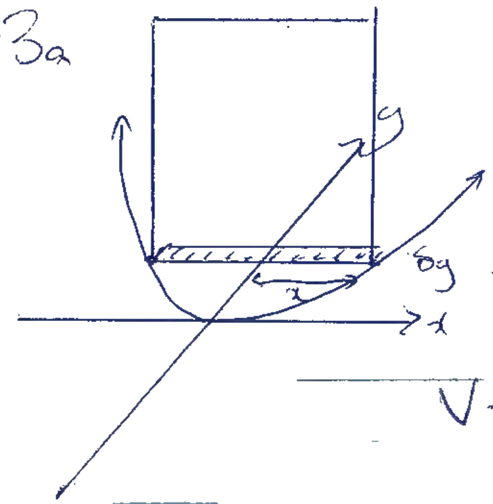
ii. $\displaystyle\int_2^4 \left(\frac{8}{3(x+4)} - \frac{5}{3(x+1)}\right) dx$

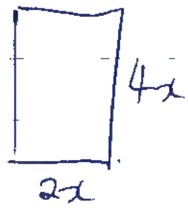
$= \dfrac{1}{3}\Big[8\ln(x+4) - 5\ln(x+1)\Big]_2^4$

$= \dfrac{1}{3}\left(8\ln 8 - 5\ln 5 - 8\ln 6 + 5\ln 3\right)$

$= -0.08422384641$

13a





$A = 8x^2$

$= 8(4y)$

$= 32y$

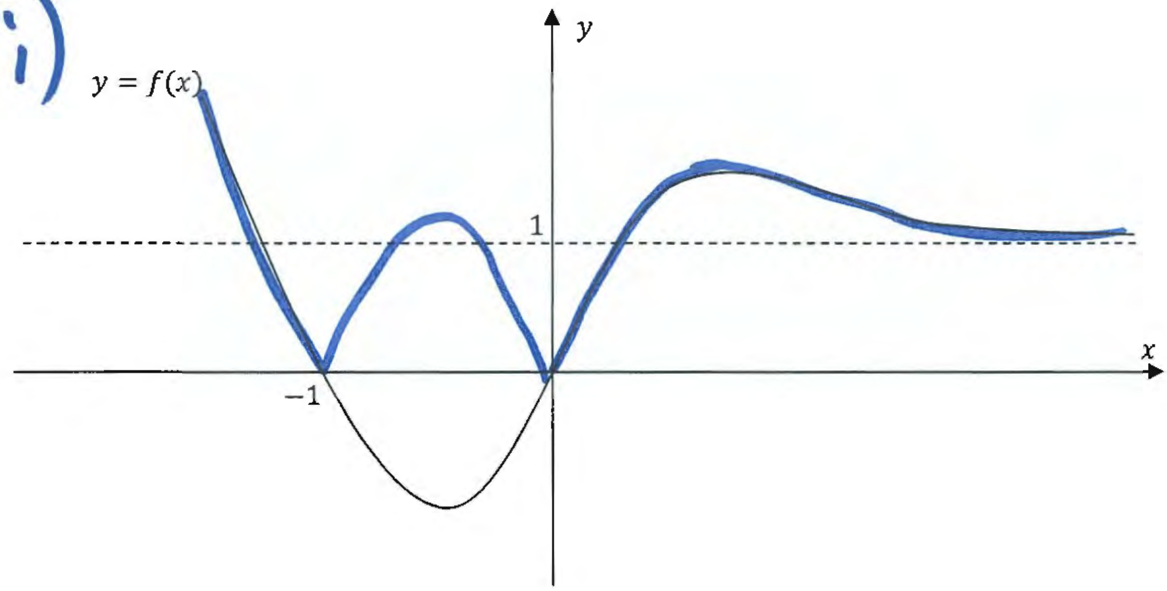
$\delta V = 32y.\delta y$

$V = \displaystyle\lim_{\delta y \to 0} \sum_{y=0}^{3} 32y\,\delta y$

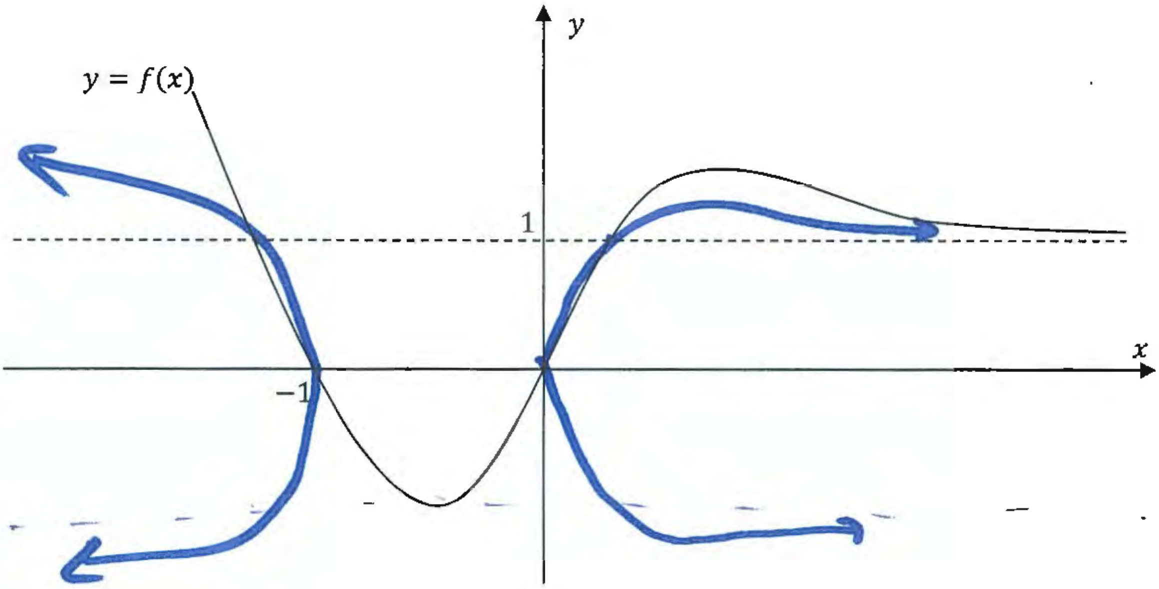
$= 32\displaystyle\int_0^3 y\,dy$

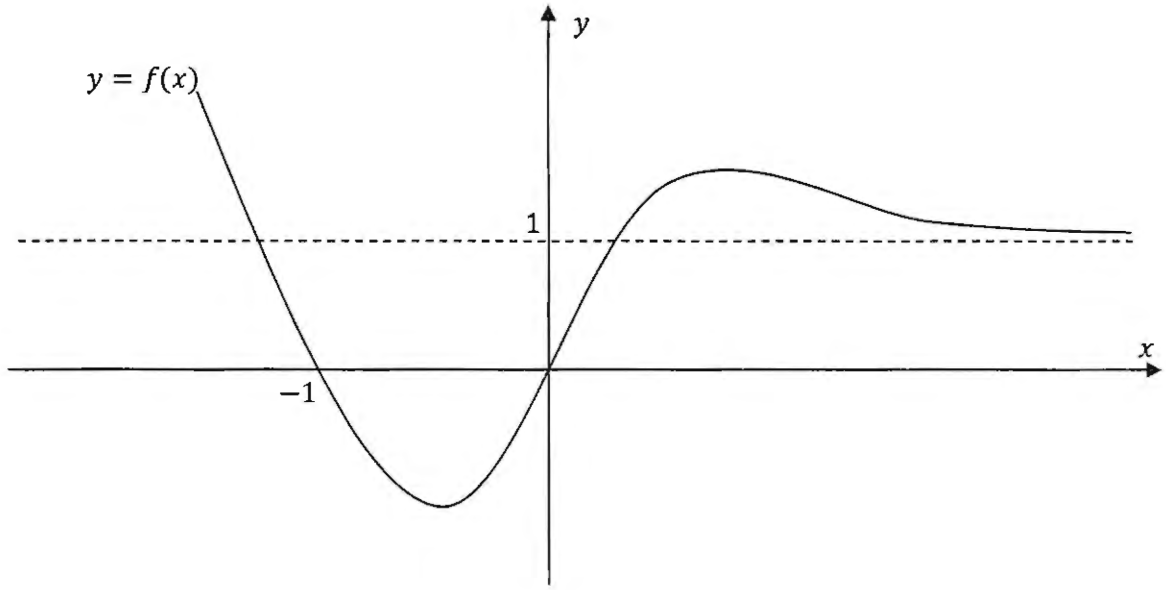
$= 32\left[\dfrac{y^2}{2}\right]_0^3 = 32\left(\dfrac{9}{2} - \dfrac{0}{2}\right) = 144 \text{ units}^3$

13b. i)

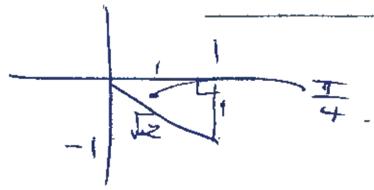


ii)





c) $z^3 = 1 - i$



$\therefore r^3 \operatorname{cis} 3\theta = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$

equating mod & arg

$r^3 = \sqrt{2}$, $\operatorname{cis} 3\theta = \operatorname{cis}\left(-\frac{\pi}{4}\right)$

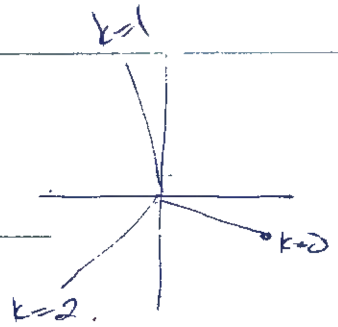
$r = \sqrt[6]{2}$ $\quad 3\theta = -\frac{\pi}{4} + 2k\pi$

$\theta = \frac{(8k-1)\pi}{12}$

for $k=0$, $\theta = -\frac{\pi}{12}$

$k=1$, $\theta = \frac{7\pi}{12}$

$k=2$, $\theta = \frac{15\pi}{12} = -\frac{9\pi}{12}$



$\therefore z = \sqrt[6]{2} \operatorname{cis}\left(-\frac{\pi}{12}\right), \sqrt[6]{2} \operatorname{cis}\left(\frac{7\pi}{12}\right), \sqrt[6]{2} \operatorname{cis}\left(-\frac{9\pi}{12}\right)$

d) i) $I_n = \int x^n e^{ax} d$

let $u = x^n$ $\quad dv = e^{ax} dx$

$du = nx^{n-1} dx$ $\quad v = \frac{e^{ax}}{a}$

$$I_n = x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot nx^{n-1} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

ii) let $I_3 = \int_0^1 x^3 e^{2x} dx$

$$= \left[\frac{x^3 e^{2x}}{2}\right]_0^1 - \frac{3}{2} I_2$$

$$= \frac{1^3 e^{2(1)}}{2} - \frac{0^3 e^{2(0)}}{2} - \frac{3}{2} I_2$$

$$= \frac{e^2}{2} - \frac{3}{2} I_2$$

$$I_2 = \left[\frac{x^2 e^{2x}}{2}\right]_0^1 - \frac{2}{2} I_1$$

$$= \frac{e^2}{2} - I_1$$

$$I_1 = \left[\frac{x e^{2x}}{2}\right]_0^1 - \frac{1}{2} I_0$$

$$= \frac{e^2}{2} - \frac{1}{2} I_0$$

$$I_0 = \int_0^1 e^{2x} dx$$

$$= \left[\frac{e^{2x}}{2}\right]_0^1$$

$$= \frac{e^2}{2} - \frac{1}{2}$$

$$= \frac{e^2 - 1}{2}$$

$$\therefore I_1 = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2-1}{2}\right)$$

$$= \frac{e^2+1}{4}$$

$$I_2 = \frac{e^2}{2} - \frac{e^2+1}{4}$$

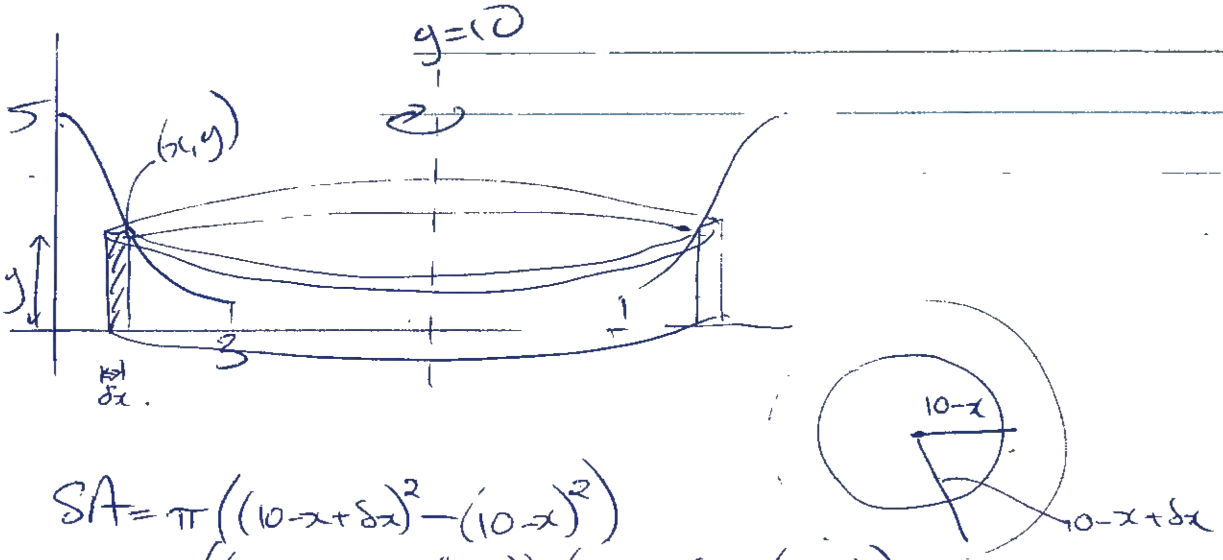
$$= \frac{e^2-1}{4}$$

$$I_3 = \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2-1}{4}\right)$$

$$= \frac{e^2+3}{4}$$

14a

i)



$$\delta A = \pi\left((10-x+\delta x)^2-(10-x)^2\right)$$
$$= \pi\left((10-x+\delta x-(10-x))\right)\left(10-x+\delta x+(10-x)\right)$$
$$= \pi\ \delta x\, 2(10-x+\delta x)$$
$$= 2\pi\,(10-x)\,\delta x \qquad \delta x^2 \doteq 0$$

$$\delta V = 2\pi\,(10-x)\delta x \cdot y$$
$$= 2\pi\,(10-x)\delta x \cdot \frac{5}{x^2+1}$$
$$V = \lim_{\delta x\to 0}\sum_{x=0}^{3}\frac{10\pi(10-x)\,\delta x}{x^2+1}$$
$$= 10\pi\int_0^3 \frac{10-x}{x^2+1}\,dx$$

ii)

$$V = 100\pi\int_0^3\frac{dx}{x^2+1} - \frac{10\pi}{2}\int_0^3\frac{2x}{x^2+1}\,dx$$
$$= 100\pi\left[\tan^{-1}x\right]_0^3 - 5\pi\left[\ln|x^2+1|\right]_0^3$$
$$= 100\pi\tan^{-1}3 - 5\pi\ln 10$$
$$= 356.2303802$$

b) i. $xy = 9$

$\frac{d}{dx}$: $x\frac{dy}{dx} + y = 0$

$$\frac{dy}{dx} = -\frac{y}{x}$$

at P, $m = \dfrac{-\frac{3}{p}}{3p}$

$$= -\frac{1}{p^2}$$

by point-gradient

$$y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$$

$$p^2y - 3p = -x + 3p$$

$\therefore x + p^2y - 6p = 0$ is equation of tangent at P — i

ii. Similarly, tangent at Q.

$x + q^2y - 6q = 0$ — ii

ii − i $(q^2 - p^2)y - (6q - 6p) = 0$

$$y = \frac{6\cancel{(q-p)}}{\cancel{(q-p)}(q+p)}$$

$$= \frac{6}{q+p}$$

Sub into ii

$$x + q^2\frac{6}{q+p} - 6q = 0$$

$$x = \frac{-6q^2 + 6q(q+p)}{q+p}$$

$$= \frac{6pq}{q+p}$$

$\therefore$ Intersection at T is $\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$

iii) chord PQ by two-point formula

$$y - \frac{3}{p} = \frac{\frac{3}{q} - \frac{3}{p}}{3q - 3p}(x - 3p)$$

$$= \frac{3(p-q)}{3pq(q-p)}(x - 3p)$$

$$y - \frac{3}{p} = -\frac{1}{pq}(x^2 - 3p)$$

$$pqy - 3q = -x^2 + 3p$$

at $(0,2)$

$$2pq - 3q = 0^2 + 3p$$

$$2pq = 3(p+q)$$

$$\therefore \frac{pq}{p+q} = \frac{3}{2}$$

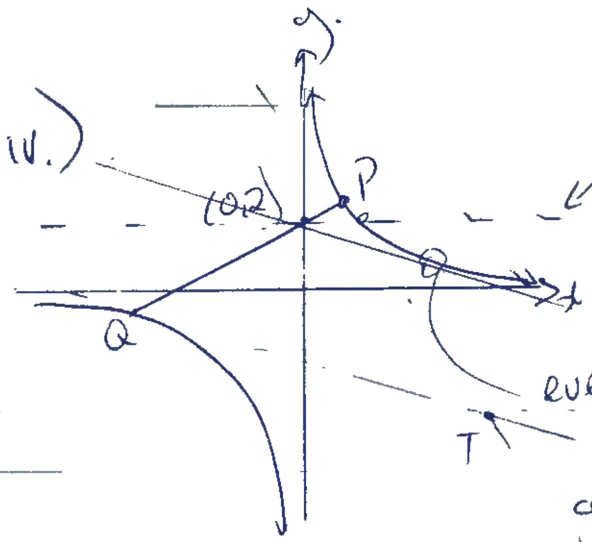
$\therefore$ Locus of T

$$x = \frac{6pq}{p+q}$$

$$= 6\left(\frac{3}{2}\right)$$

$$= 9.$$

iv.)



P cannot have $y=2$, this gives no chord. $(p \neq \frac{3}{2})$

eventually, chord becomes tangent to curve as P drops below $y=2$.

restriction: y-int. of tangent > y-int of chord.

$$\frac{6p}{p^2} > 2$$

$$\therefore \frac{6}{p} > 2$$

$$p < 3$$

hence, $y < 1$

C.

$$\left|1+\frac{1}{z}\right| \leq 1$$

$$\left|1+\frac{1}{x+iy}\right| \leq 1$$

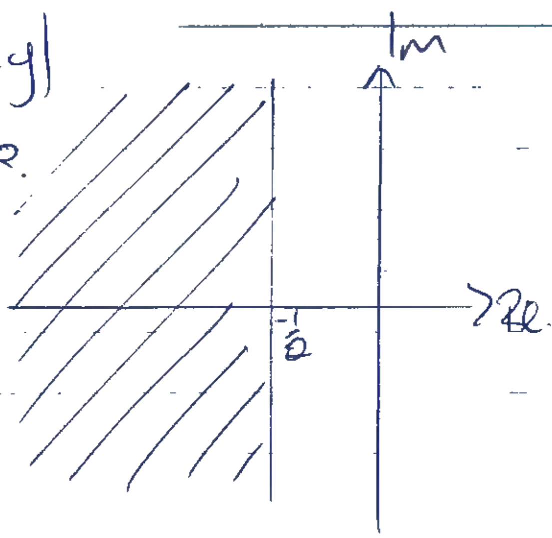
$$\left|\frac{x+1+iy}{x+iy}\right| \leq 1$$

$$|x+1+iy| \leq |x+iy|$$

squaring $\quad (x+1)^2+y^2 \leq x^2+y^2$

$$2x+1 \leq 0$$

$$x \leq -\frac{1}{2}$$



Q15 a $P(x) = x^5 + 2x^2 + mx + n.$

$P(-2) = P'(-2) = 0$

$P'(x) = 5x^4 + 4x + m$

$P(-2)$: $(-2)^5 + 2(-2)^2 - 2m + n = 0$

$\therefore 8 - 2m + n = 0$ — i

$P'(-2)$: $5(-2)^4 - 2(4) + m = 0$

$\therefore m = -72$

from i

$8 - 2(-72) + n = 0$

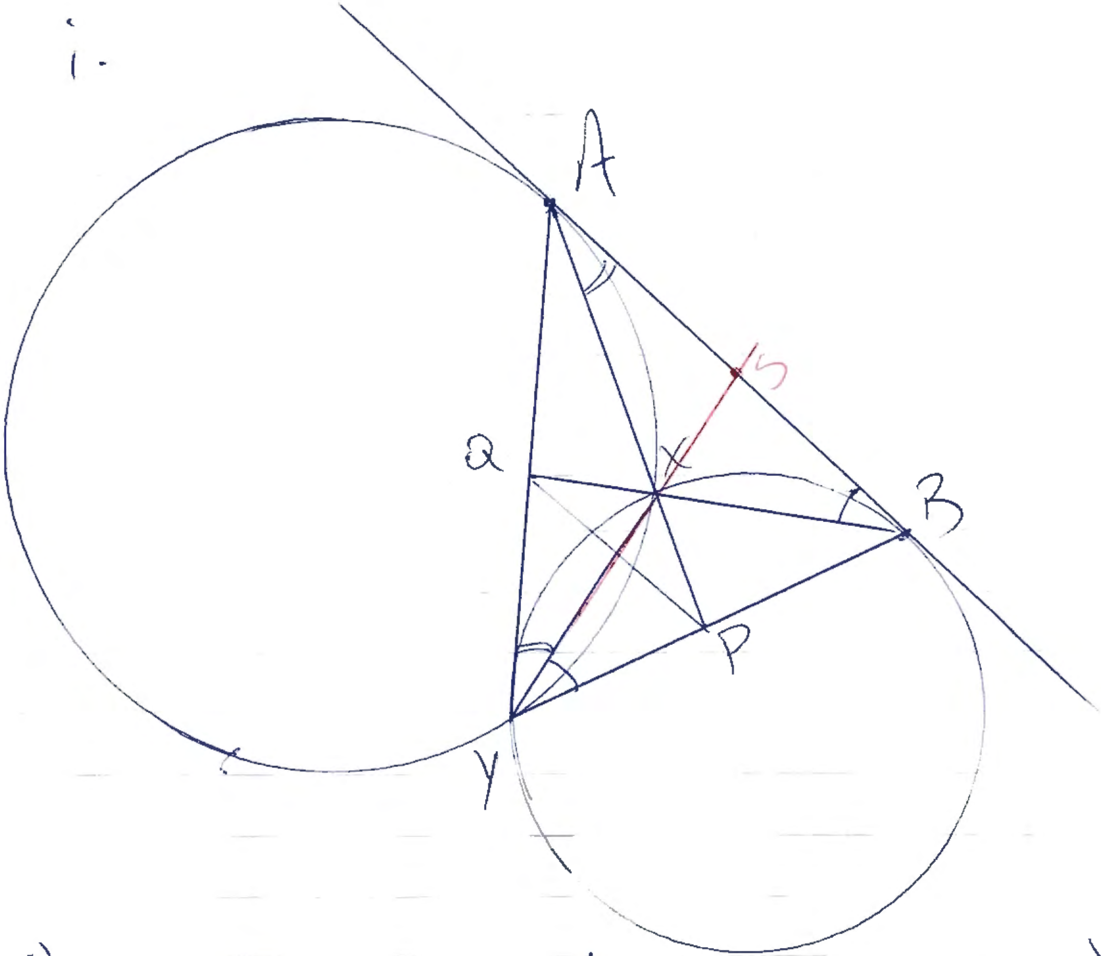
$n = 152$

product of roots: $\alpha\beta\gamma(-2)^2 = \frac{-n}{1}$

$4\alpha\beta\gamma = -152$

$\therefore \alpha\beta\gamma = -38$

b. i.



ii). $\angle ABQ = \angle BYX$ (angle in alternate segment)

similarly,

$\angle BAP = \angle AYX$

$\angle BAP + \angle ABQ + \angle AXB = 180°$ (angle sum of triangle AXB)

$\therefore \underline{\angle AYX + \angle BYX} + \angle QXP = 180°$ (vertically opp. $\angle AXB = \angle QXP$)

$\therefore \angle QYP \quad + \angle QXP = 180°$

QXPY is a cyclic quad (opp angles supplementary)

iii) $\angle PYX = \angle XQP$ (angles in same segment, circle XQYP)

$\therefore \angle ABQ = \angle XQP$

PQ // BA (alternate angles equal)

iv) Extend YX to meet AB at S.

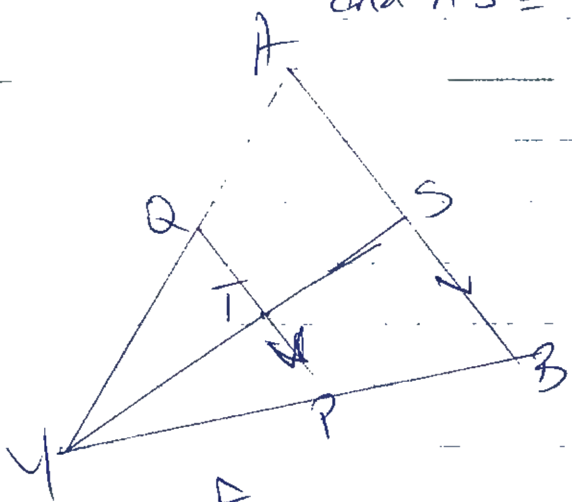
$AS^2 = SX.SY$. (square of tangent equals product of secant

similarly

$BS^2 = SX.SY$

$\therefore AS^2 = BS^2$

and $AS = BS$, hence S is the midpoint of AB)



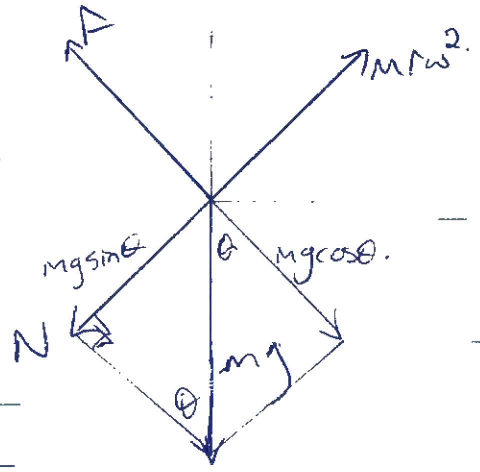
$$\frac{QT}{PT} = \frac{AS}{BS} \quad \text{(ratio of intercepts)}$$

$$= 1$$

$\therefore QT = PT$

and XY bisects PQ.

c)



perpendicular: $F - mg\cos\theta = 0$

parallel: $N + mg\sin\theta - mr\omega^2 = 0$

$F = mg\cos\theta$

$N = mr\omega^2 - mg\sin\theta$

$$\frac{F}{N} = \frac{mg\cos\theta}{m(r\omega^2 - g\sin\theta)}$$

$$= \frac{g\cos\theta}{r\omega^2 - g\sin\theta}$$

Q16

a

$$\ddot{x} = -g - kv$$

$$v\frac{dv}{dx} = -g - kv$$

$$\int -\frac{v\,dv}{g+kv} = \int dx$$

$$x = -\int \frac{v\,dv}{g+kv}$$

$$= -\frac{1}{k}\int \frac{kv\,dv}{g+kv}$$

$$= -\frac{1}{k}\int \frac{g+kv-g}{g+kv}\,dv$$

$$= -\frac{1}{k}\int \left(1 - \frac{g}{g+kv}\right)dv$$

$$= -\frac{1}{k}\left(v - \frac{g}{k}\ln(g+kv)\right) + C$$

$$= -\frac{v}{k} + \frac{g}{k^2}\ln(g+kv) + C$$

for $x=0$, $v=V$

$$C = \frac{V}{k} - \frac{g}{k^2}\ln(g+kV)$$

So $x = -\frac{v}{k} + \frac{V}{k} + \frac{g}{k^2}\ln(g+kv) - \frac{g}{k^2}\ln(g+kV)$

for max height, $v=0$

$$H = \frac{V}{k} + \frac{g}{k^2}\ln(g) - \frac{g}{k^2}\ln(g+kV)$$

$$= \frac{V}{k} - \frac{g}{k^2}\ln\left(\frac{g+kV}{g}\right)$$

$$= \frac{V}{k} - \frac{g}{k^2}\ln\left(1+\frac{kV}{g}\right)$$

b) i) $2p+1 < 2p+2$

$$\frac{1}{2p+1} > \frac{1}{2p+2}$$

$$\frac{1}{2p+1} + \frac{1}{2p+2} > \frac{1}{2p+2} + \frac{1}{2p+2}$$

$$> \frac{2}{2p+2}$$

$$> \frac{1}{p+1}$$

ii) $\psi(m): \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m} \geq \frac{37}{60}$

Prove true for $m=3$

$$LHS = \frac{1}{3+1} + \frac{1}{3+2} + \frac{1}{3+3}$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

$$= \frac{37}{60} = RHS$$ $\therefore$ true for $m=3$

Assume true for $n=k$.

$$\therefore \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k} \geq \frac{37}{60}$$

Prove true for $n=k+1$

i.e.

i.e. $\frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{2k+2} \geq \frac{37}{60}$

$$LHS = \frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$$= \frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\dots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}-\frac{1}{k+1}$$

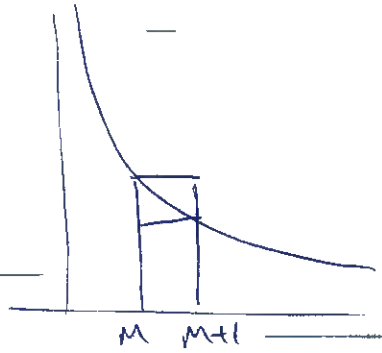
$$\geq \frac{37}{60}+\frac{1}{2k+1}+\frac{1}{2k+2}-\frac{1}{k+1} \quad \text{by induction hypothesis}$$

$$\geq \frac{37}{60}+\frac{1}{k+1}-\frac{1}{k+1} \quad \text{by part (b i)}$$

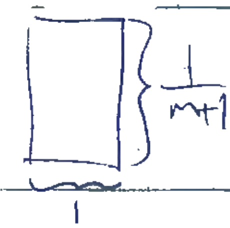
$$\geq \frac{37}{60}$$

$\therefore$ statement is true, by mathematical induction.

iii) low rectangle $< \int_m^{m+1} \frac{1}{t}\,dt <$ high rectangle



area of low rectangle $= 1 \cdot \frac{1}{m+1}$



$$\therefore \int_m^{m+1} \frac{1}{t}\,dt > \frac{1}{m+1}$$

iv) $$\int_m^{2m} \frac{1}{t}\,dt = \int_m^{m+1} \frac{1}{t}\,dt + \int_{m+1}^{m+2} \frac{1}{t}\,dt + \dots + \int_{2m-1}^{2m} \frac{1}{t}\,dt$$

$$> \frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{2m} \quad \text{from iii)}$$

$$> \frac{37}{60} \quad \text{from ii)}$$

Also $\displaystyle\int_n^{2n} \frac{1}{t}\,dt = \Big[\ln t\Big]_n^{2n}$

$$= \ln 2n - \ln n$$

$$= \ln\left(\frac{2n}{n}\right)$$

$$= \log_e 2$$

$$\therefore \log_e 2 > \frac{37}{60}$$